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# Note

## A Note on Quasi-Normed Convolution Algebras of Entire Analytic Functions of Exponential Type

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Some quasi-Banach spaces of entire analytic functions of exponential type are convolution algebras.

### 1. INTRODUCTION

This note is more or less an appendix to [4] and [5]. We shall use the notations and the results of these two papers.

Let $S'$ be the usual space of tempered distributions in Euclidean $n$-space $R_n$. The Fourier transform in $S'$ and its inverse are denoted by $F$ and $F^{-1}$, respectively. Let $\Omega$ be a bounded $C^\infty$-domain in $R_n$ (or more generally, a bounded domain with restricted cone property; a definition of this property may be found, for instance, in [5]). If $\kappa(x)$ is a nonnegative weight function in $R_n$ and if $0 < p \leqslant 1$, then

$$L_p^{\Omega,\kappa} = \left\{ f \mid f \in S', \operatorname{supp} Ff \subset \bar{\Omega}, \|f\|_{L_p\kappa} = \left(\int_{R_n} \kappa^p(x) \mid f(x)|^p \, dx\right)^{1/p} < \infty \right\}.$$

There is a corresponding definition for $1 < p \leqslant \infty$, but here we are only interested in the cases $0 < p \leqslant 1$. (The more general framework of ultra-distributions $(S_a)'$ used in [4] instead of $S'$ is not necessary here: All the spaces under consideration will be subspaces of $L_1$.) By the Paley–Wiener–Schwartz theorem an element of $L_p^{\Omega,\kappa}$ is the restriction to $R_n$ of an entire analytic function of exponential type of $n$ complex variables. We shall not describe the most general class of admissible weight functions $\kappa(x)$. Apart from some



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remarks in Section 3 we shall be concerned with the following three typical weight functions:

$$\kappa(x) = |x|^{\alpha}, \qquad \alpha \geqslant 0, \tag{1}$$

$$\kappa(x) = \prod_{j=1}^{n} |x_j|^{\alpha_j}, \qquad \alpha_j \geqslant 0, \tag{2}$$

$$\kappa(x) = e^{\beta |x|^{\gamma}}, \qquad \beta \geqslant 0, \quad 0 \leqslant \gamma < 1. \tag{3}$$

The problem treated in this note is the following: Assume that $f$ and $g$ are elements of $L_p^{\Omega,\kappa}$. Is the convolution $f * g$ meaningful, and is $f * g$ again an element of $L_p^{\Omega,\kappa}$? The theorem in Section 2 gives an affirmative answer provided that $0 < p \leqslant 1$. In Section 3 there will be given some remarks mostly concerned with possible generalizations.

All unimportant positive numbers will be denoted with the same letter $c$.

## 2. Theorem

If $\kappa(x)$ is one of the weight functions (1)–(3), then we set

$$\rho(x) = 1 + |x|^{\alpha}, \qquad \rho(x) = \prod_{j=1}^{n} (1 + |x_j|^{\alpha_j}) \tag{4}$$

in the first two cases respectively, and $\rho(x) = \kappa(x)$ in the third case ($\alpha$ and $\alpha_j$ have the same meaning as in (1) and (2)). In [4, Theorem 4.1] (see also Examples in [4, Sect. 2.3]) it was shown that

$$L_p^{\Omega,\kappa} = L_p^{\Omega,\rho}. \tag{5}$$

But $\rho(x) \geqslant 1$. Consequently, $L_p^{\Omega,\kappa} \subset L_p^{\Omega}$ where $L_p^{\Omega}$ is the corresponding space with weight function 1. Using $L_p^{\Omega} \subset L_1^{\Omega}$ for $0 < p \leqslant 1$ (essentially this is an inequality of Nikol'skij type, see, for instance, [4]) it follows that $L_p^{\Omega,\kappa}$, $0 < p \leqslant 1$, is a subspace of the convolution algebra $L_1$. In particular, the convolution $f * g$ is meaningful if $f$ and $g$ are elements of $L_p^{\Omega,\kappa}$. In an easy to understand notation we shall write $L_p^{\Omega,\kappa} * L_p^{\Omega,\kappa}$.

THEOREM. *If $\kappa(x)$ is one of the three weight functions* (1)–(3) *and if $0 < p \leqslant 1$, then there holds*

$$L_p^{\Omega,\kappa} * L_p^{\Omega,\kappa} \subset L_p^{\Omega,\kappa}, \tag{6}$$

*and there exists a positive number $c$ such that for all $f \in L_p^{\Omega,\kappa}$ and all $g \in L_p^{\Omega,\kappa}$*

$$\| f * g \|_{L_p^\kappa} \leqslant c \| f \|_{L_p^\kappa} \| g \|_{L_p^\kappa} \tag{7}$$

(*quasi-normed convolution algebras*).

*Proof.* (The proof is essentially a simple interpretation of two formulas proved in [4] and [5].) Using (5) and the above functions $\rho$, the estimate (7) can be rewritten as

$$\| F^{-1}(Fg\, Ff) \|_{L_p^\rho} \leqslant c \| g \|_{L_p^\rho} \| f \|_{L_p^\rho}, \tag{8}$$

in other words: We must show that $Fg$ is a Fourier multiplier in $L_p^{\Omega,\rho}$. The functions $\rho$ of the first two cases satisfy the conditions of Definition 1 in [5] (formula (2)). So we can apply Theorem 1(i) of [5]: Let $Q$ be an open cube with side length $d = 2\pi/h$ such that $\bar{\Omega} \subset Q$ (here $h$ is an arbitrary positive number less than a critical number $h_0$, see [4]). Let $g \in L_p^{\Omega,\rho}$ and let ($m = (m_1, \ldots, m_n)$, $m_j$ integers, $mx = \sum_{j=1}^n m_j x_j$)

$$M_m = d^{-n} \int_Q (Fg)(x)\, e^{i(2\pi/d)mx}\, dx \tag{9}$$

be the corresponding Fourier coefficients of $Fg$ (formula (15) in [5]). The lattice of all points in $R_n$ whose coordinates are integers will be denoted by $N$. If there exists a number $A$ such that for all sequences $\{a_t\}_{t\in N}$

$$\left\| \left\{ \rho(hm) \sum_{t\in N} M_{m-t} a_t \right\}_{m\in N} \right\|_{l_p} \leqslant A \| \{\rho(hm)\, a_m\}_{m\in N} \|_{l_p}, \tag{10}$$

then it follows from Theorem 1(i) in [5] that

$$\| F^{-1}(Fg\, Ff) \|_{L_p^\rho} \leqslant cA \| f \|_{L_p^\rho} \tag{11}$$

for all $f \in L_p^{\Omega,\rho}$. Here $c$ is independent of $f$ and $g$. As mentioned in [5, 3.3] (see also [5, Proposition 3]) the argument in [5] holds true as well in the case of the thrid function $\rho$. In particular, formulas (10) and (11) are valid in this case, too. The above three functions $\rho$ satisfy

$$\rho(x) \leqslant c\rho(y)\, \rho(x-y) \qquad (\text{all } x \in R_n,\ y \in R_n). \tag{12}$$

Thus (10) yields (using $0 < p \leqslant 1$)

$$\begin{aligned} \sum_{m\in N} \left| \rho(hm) \sum_{t\in N} M_{m-t} a_t \right|^p &\leqslant \sum_{m,t\in N} \rho^p(hm)\, M_{m-t}^p\, | a_t |^p \\ &\leqslant c \sum_{t\in N} \rho^p(ht)\, | a_t |^p \sum_{m\in N} \rho^p(hm-ht)\, M_{m-t}^p \\ &\leqslant c \Big( \sum_{m\in N} \rho^p(hm)\, M_m{}^p \Big) \Big( \sum_{t\in N} \rho^p(ht)\, | a_t |^p \Big). \end{aligned} \tag{13}$$

Consequently, the number $A$ in (11) can be estimated by

$$c\left(\sum_{m\in N} \rho^p(hm)\, M_m{}^p\right)^{1/p}. \tag{14}$$

On the other hand, the integration over $Q$ in (9) can be extended to $R_n$ (the $L_\infty$-function $Fg$ vanishes outside of $Q$). This proves

$$M_m = c[F^{-1}(Fg)]\left[\frac{2\pi}{d}\, m\right] = cg(hm).$$

Using the fact that

$$\|\{\rho(hm)\, g(hm)\}_{m\in N}\|_{l_p}$$

is an equivalent quasi-norm in $L_p^{\Omega,\rho}$ (see [4, Theorem 4.1 and formula (29)]), it follows that

$$\| F^{-1}(Fg\, Ff)\|_{L_p{}^\rho} \leqslant c\, \| g\, \|_{L_p{}^\rho} \|f\|_{L_p{}^\rho}.$$

This proves (7). Obviously, $\operatorname{supp} F(f * g) = \operatorname{supp} Ff \cdot Fg \subset \bar{\Omega}$. This completes the proof of the theorem.

## 3. Remarks

*Remark* 1. The proof is very simple (using [4] and [5]). But when the two papers [4, 5] were written the author did not notice this possibility. The case $\kappa(x) \equiv 1$ seems to be known: It was used recently by Jawerth [1] with a reference to a forthcoming book by Peetre [2, Chap. 11].

*Remark* 2. We used bounded $C^\infty$-domains $\Omega$ (or bounded domains with restricted cone property) and the special weight functions (1)–(3). Both these hypotheses can be weakened. Smootheness conditions for the domains are used in [5] in the later parts. Theorem 1(i) in [5] (which we used here) is independent of these smoothness conditions (although it was not stated explicitly, but it follows from the proof and the above assumptions). This shows that the above theorem remains true if $\bar{\Omega}$ is an arbitrary compact set in $R_n$. The condition for the weight function $\rho(x)$ can be generalized essentially: Let $\rho(x)$ be a Borel measurable positive function in $R_n$ such that (12) holds true and

$$0 < \rho(x) \leqslant ce^{\beta|x|^\gamma},$$

where $c$ and $\beta$ are appropriate positive numbers and $0 \leqslant \gamma < 1$. This is an admissible $\rho$-function in the sense of Definition 2.1 in [4]. Assume additionally $\rho(x) = \rho(-x)$. Then (12) yields

$$\rho^2(y) \geqslant (1/c)\,\rho(0) > 0.$$

In particular, the corresponding space $L_p^{\Omega,\rho}$, $0 < p \leqslant 1$, is again a subset of $L_1$. Then all the above considerations remain true: *The above theorem is valid for the corresponding spaces* $L_p^{\Omega,\rho}$ $(0 < p \leqslant 1$, $\Omega$ *being an arbitrary compact set in* $R_n)$. Furthermore, the weight function $\rho(x)$ may be replaced by more general weights $\kappa(x)$ in the sense of Theorem 4.1 in [4].

*Remark* 3. There is another possibility for generalizations of the above theorem. Recently, Stöckert [3] considered Plancherel–Polya–Nikol'skij inequalities in weighted spaces with mixed $L_p$-norms or quasi-norms, respectively. That means that $\|f\|_{L_p}$ is replaced by

$$\left(\int_{R_1}\left(\int_{R_1}\cdots\left(\int_{R_1}|f(x_1,\ldots,x_n)|^{p_n}\,dx_n\right)^{(p_{n-1})/p_n}\cdots dx_2\right)^{p_1/p_2}dx_1\right)^{1/p_1}.$$

The results are similar to corresponding results in [4]. It seems to be possible to prove corresponding multiplier theorems and to obtain results similar to the above theorem.

*Remark* 4. Estimates of type (7) seem to be useful in the theory of function spaces, see [1, 2] for $\kappa(x) \equiv 1$. In this connection the dependence of the number $c$ in (7) on $\Omega$ is of interest. Let

$$\Omega = K_r = \{y \mid |y| \leqslant r\}, \qquad 0 < r < \infty.$$

Let $\kappa(x)$ be the weight function from (1) or from (2). If $f(x) \in L_p^{K_r,\kappa}$, then $f(x/r) \in L_p^{K_1,\kappa}$. Similarly for $g(x) \in L_p^{K_r,\kappa}$. Apply (7) to $f(x/r)$ and $g(x/r)$. A simple transformation of coordinates yields $(0 < p \leqslant 1)$

$$\|f * g\|_{L_p\kappa} \leqslant c r^{\alpha+n(1/p-1)} \|f\|_{L_p\kappa} \|g\|_{L_p\kappa}; \qquad \kappa = |x|^\alpha;$$

and

$$\|f * g\|_{L_p\kappa} \leqslant c r^{\alpha_1+\cdots+\alpha_n+n(1/p-1)} \|f\|_{L_p\kappa} \|g\|_{L_p\kappa}; \qquad \kappa = \prod_{j=1}^{n} |x_j|^{\alpha_j},$$

where $c$ is independent of $r$.

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